AP Calculus AB
Test: Separable Differential Equations
Instructor Version

Free-Response Question 1 (with calculator)
You may use your calculator for this question.
(When you finish this question, if your test is proctored, ask your proctor for the next section. If you're taking the unproctored version of the test, continue with the next question.)

1. Let $C(t)$ be the number of cougars on an island at time t years (where $t \geq 0$). The number of cougars is increasing at a rate directly proportional to $3500 - C(t)$. Also, $C(0) = 1000$, and $C(5) = 2000$.

A. Find $C(t)$ as a function of t only.

Answer:

$\frac{dC}{dt} = k(3500 - C)$ (1 point)

$\frac{dC}{3500 - C} = k\,dt$ (1 point)

$-\ln|3500 - C| = kt + b$ (2 points)

$\ln|3500 - C| = -(kt + b)$

$3500 - C = Ae^{-kt}$

$C = 3500 - Ae^{-kt}$ (1 point)

$1000 = 3500 - Ae^{-k(0)} \Rightarrow A = 2500$ (1 point)

$2000 = 3500 - 2500e^{-k(5)} \Rightarrow k \approx 0.102$ (1 point)

$C = 3500 - 2500e^{-0.102t}$ (1 point)

B. Calculate $C(10)$.

Answer: $C(10) \approx 3500 - 2500e^{-0.102 \cdot 10} \geq \approx 2598$ cougars (2 points)

1 point: approximation symbol
1 point for answer (Note: 2599 is also acceptable)

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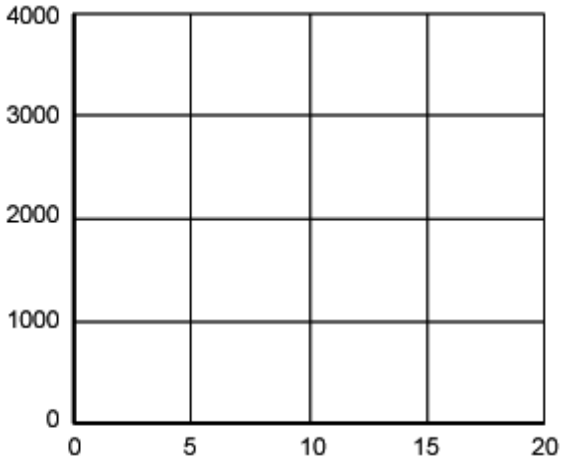
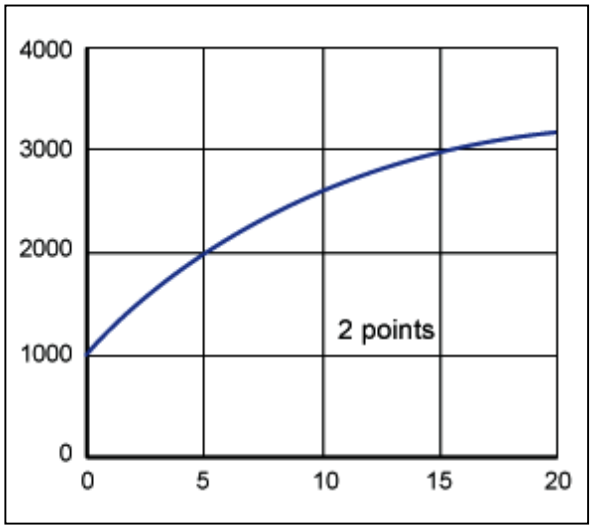
C. Find $\lim_{t \to \infty} C(t)$, and explain its meaning.

Answer: $\lim_{t \to \infty} C(t)$ = 3500, which means that as time continues forever, the cougar population will approach 3500 cougars, but never go higher.

2 points for limit
1 point for interpretation

D. On the axes provided, draw a graph showing the number of cougars as a function of time:





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Free-Response Question 2 (no calculator)
You may not use your calculator for this question.

2. The expression $\frac{dy}{dx} = x\sqrt[3]{y}$ gives the slope at any point on the graph of the function $f(x)$ where $f(2) = 8$.

A. Write the equation of the tangent line to $f(x)$ at point (2, 8).

Answer: $y - 8 = 4\ (x-2)$ or $y = 4x$

1 point for finding the slope
2 points for tangent line equation in any form.

B. Write an expression for $f(x)$ in terms of x.

Answer:

$$\frac{dy}{dx} = x\sqrt[3]{y}$$

$$y^{-1/3}\ dy = x\ dx \qquad \text{(1 point)}$$

$$\frac{3}{2}\ y^{2/3} = \frac{1}{2}x^2 + C_1 \qquad \text{(1 point)}$$

$$y = \left(\frac{1}{3}x^2 + C_2\right)^{3/2}$$

$$8 = \left(\frac{1}{3}2^2 + C_2\right)^{3/2} \qquad \text{(1 point)}$$

$$\frac{8}{3} = C_2$$

$$y = f(x) = \left(\frac{1}{3}x^2 + \frac{8}{3}\right)^{3/2} \qquad \text{(1 point)}$$

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C. What is the domain of $f(x)$?

Answer: The radicand must be greater than or equal to zero, so $\left(\frac{1}{3}x^2 + \frac{8}{3}\right) \geq 0$.

This is true for all real numbers, which is the domain. (2 points)

Note: If students are correctly using a wrong answer from part B, they should get full credit.

D. What is the maximum value of $f(x)$?

Answer: Use the Extreme Value Theorem to examine $\frac{dy}{dx} = x\sqrt[3]{y} = 0$. The minimum will occur at $x = 0$. A quick check of the slopes on either side of $x = 0$ shows that:

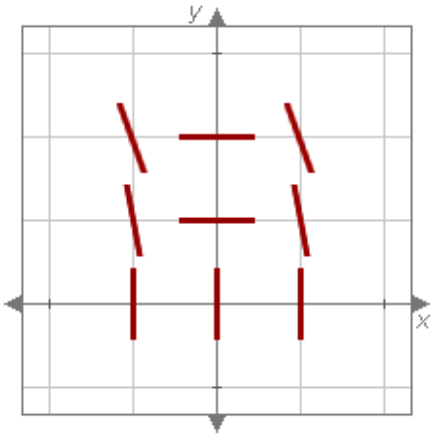
x	$x < 0$	$x = 0$	$x > 0$
$f'(x)$	**negative**	**0**	**positive**
$f(x)$	**decreasing**	**minimum**	**increasing**

so the minimum value is $f(0) = \left(\frac{8}{3}\right)^{3/2}$

(4 points): 1 point for setting derivative equal to zero
1 point for determining critical points
1 point for checking the slope
1 point for the maximum value

E. Using the axes provided, sketch a slope field for the given differential equation at the nine points indicated. (2 points)

Answer:



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